Can Unbiased Estimators be Absurd?

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Overview

We use the concept of parameter space and introduce the concept of potential absurdity to examine an undesirable property of an unbiased estimator in an undergraduate setting.

Motivation

- Undergraduate texts emphasize unbiasedness when estimating parameters of random variables.
- Good estimators are unbiased, but are all unbiased estimators good?

Unbiased Estimation

Unbiasedness:

- A statistic, $\hat{\theta}$, is said to be an unbiased estimator for a parameter θ, if E($\hat{\theta}$) = θ.

  e.g., $E(\bar{X}) = \mu$

How Do We Find An Unbiased Estimator?

- Suppose X is a continuous random variable with the probability density function,

$$f(x,\theta) = \theta x + \frac{1}{2}, \quad -1 \le x \le 1.$$

- First, $\mu = \int_{-1}^{1} x\left(\theta x + \frac{1}{2}\right) dx = \frac{2}{3}\theta,$ which is a linear function of θ.
- Then, $\theta = \frac{3}{2}\mu$.
- Now, we replace μ with $\bar{X}$, and we get that $\hat{\theta} = \frac{3}{2}\bar{X}$ is an unbiased estimator for θ.

Numerical Example

- Suppose we have a sample of size 5 from the above distribution, as follows:
  $\{-0.2,\ -0.1,\ 0.6,\ 0.8,\ 0.9\}$
- Then, $\bar{X} = 0.4$
- So, $\hat{\theta} = \frac{3}{2} \cdot 0.4 = 0.6$
- This is the scope of most undergraduate content on this matter.

Definition: Parameter Space

- For, $X \sim f(x, \theta)$, the set of all possible values of θ is called the parameter space of θ.
- Denoted:

Ω_θ = The parameter space of θ

Examples

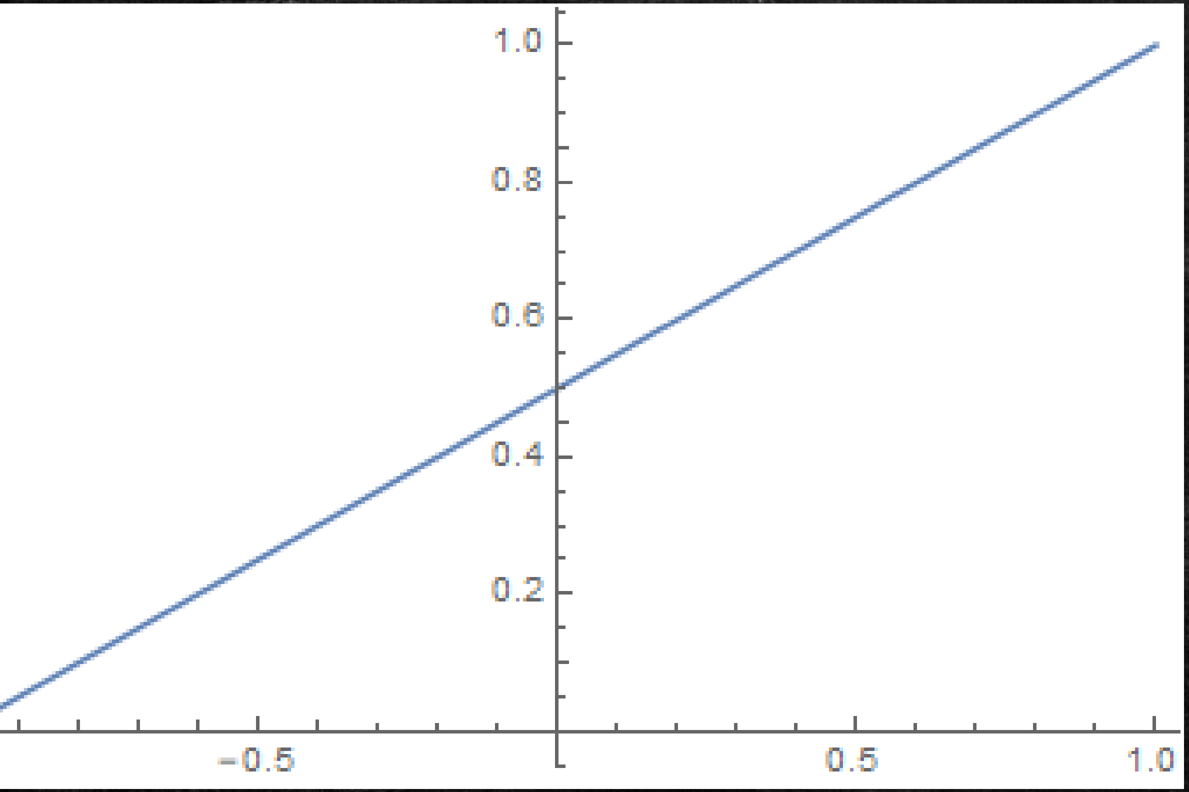
- $X \sim bin(n, p) \Rightarrow \Omega_p = [0,1]$
- $X \sim \exp(\lambda) \Rightarrow \Omega_\lambda = \mathbb{R}^+$
- $X \sim \text{Pois}(\lambda) \Rightarrow \Omega_\lambda = \mathbb{R}^+$

Another Example: Finding the Parameter Sp

- Recall our previous example, $f(x,\theta) = \theta x + \frac{1}{2}, \quad -1 \leq x \leq 1$
- What is the parameter space for θ?
- Being a probability density function, $f(x,\theta) \geq 0$, for $-1 \leq x \leq 1$

Another Example: Finding the Parameter Sp

$$f(x,\theta) = \theta x + \frac{1}{2}, \qquad -1 \le x \le 1$$



- $f(-1) \geq 0$

$\Rightarrow \ -\theta + \frac{1}{2} \geq \ 0 \ \Rightarrow \theta \leq$

- $f(1) \geq 0$

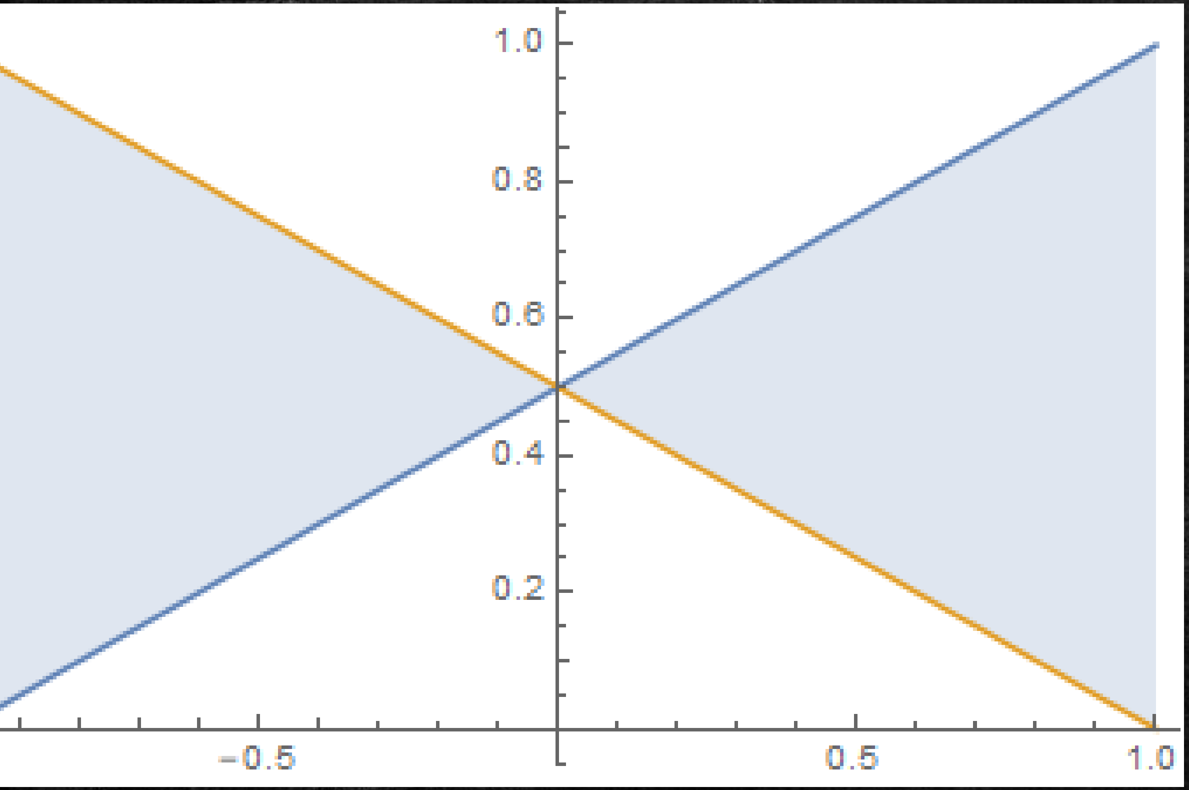
$\Rightarrow \theta + \frac{1}{2} \geq \ 0 \ \Rightarrow \theta \geq -$

- Putting the two cond together gives us:

$\Omega_\theta = [-0.5, 0.5\,]$

Another Example: Finding the Parameter Sp

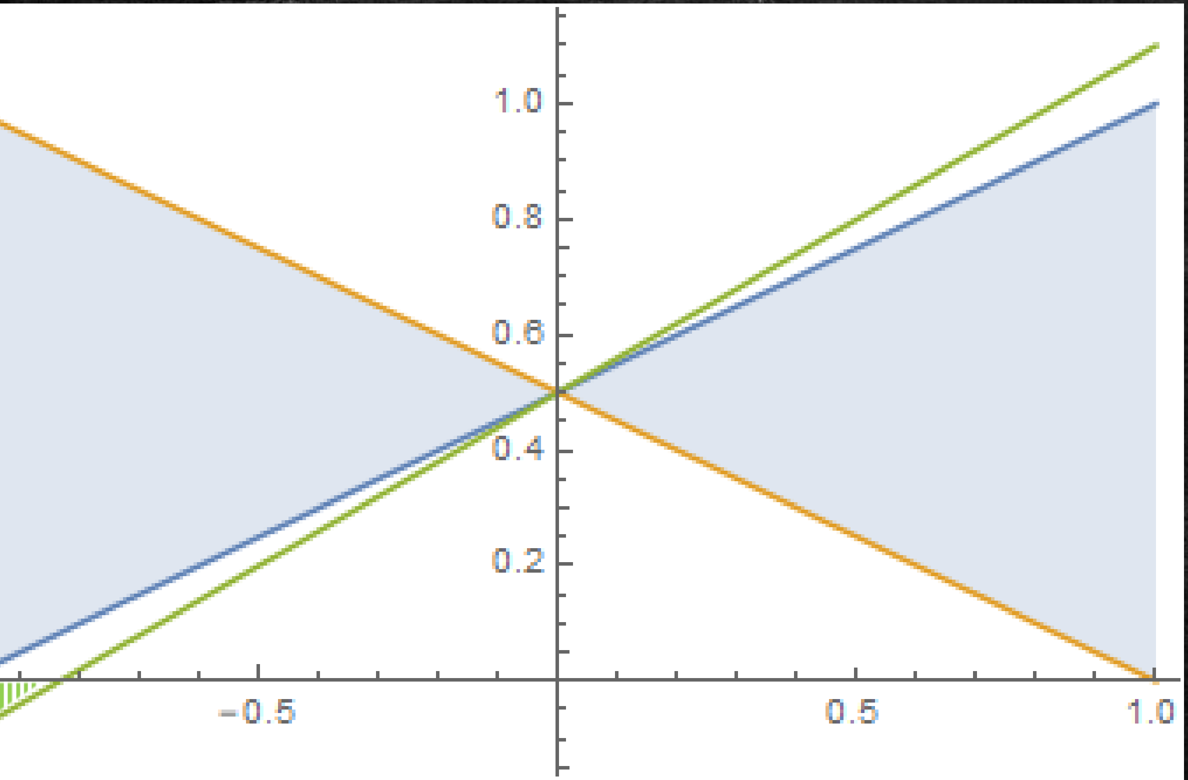
$$f(x,\theta) = \theta x + \frac{1}{2}, \qquad -1 \le x \le 1$$



- The shaded area show possible lines generat values of θ in Ω_θ.
- Now, recall that we fo $\hat{\theta} = \frac{3}{2}\bar{X}$ to be an unbi estimator for θ.
- Furthermore, we gene an estimate of $\hat{\theta} = 0.6$

However…

$$f(x, \theta) = \theta x + \frac{1}{2}, \qquad -1 \leq x \leq 1$$



- The unbiased estimator this estimate:

  $\hat{\theta} = 0.6$, given by the gr

- The estimate is outside

  $$\Omega_\theta = [-0.5, 0.5]$$

- The green line results in probability density.

- This makes the estimate even though it is unbias

Definition: Potential Absurdity

- An estimate of θ which is not in Ω_θ is called absurd

Definition:

- An unbiased estimator, $\hat{\theta}$, for θ is called "Potentially Absurd" for θ if there is a probability of an estimate outside Ω_θ,

i.e., $$P\left(\hat{\theta} \notin \Omega_\theta\right) > 0$$

… And It Could Be Worse

- Let $X =$ the number of customers entering a store during a ten minute period.
- Then we can model $X \sim Pois(\lambda)$
- Let $\theta = P(no\ arrivals\ in\ 1\ hour)$
- Let W $=$ number of arrivals in one hour
  - By additivity, W $= \sum_{i=1}^{6} X_i \sim Pois(6\lambda)$

So, $\theta = P(W = 0) = e^{-6\lambda}$

A Quirky Unbiased Estimator

- Since θ is a probability, then $\Omega_\theta = [0, 1]$.
- Suppose we have a sample of size 2, $\{X_1,\ X_2\}$
- Now consider the estimator for θ: $\hat{\theta} = (-2)^{X_1+X_2}$
  - Let Y $= X_1 + X_2 \sim Pois(2\lambda)$
  - Then E($(-2)^Y$) $= \sum_{y=0}^{\infty} (-2)^y\, e^{-2\lambda}\, \frac{(2\lambda)^y}{y!}$

$$= e^{-2\lambda} \sum_{y=0}^{\infty} \frac{(-4\lambda)^y}{y!} = e^{-2\lambda} e^{-4\lambda} = e^{-6\lambda}$$

- Which means that $\hat{\theta} = (-2)^{X_1+X_2}$ is an unbiased estimator for θ.

But Is $\hat{\theta} = (-2)^{X_1+X_2}$ A Good Estimator?

- For instance, if $X_1 = 3$ and $X_2 = 7$, then $\hat{\theta} = (-2)^{3+7} = 1024$
- In fact, the only way $\hat{\theta}$ will give an acceptable estimate for

  $\theta = P(no\ arrivals\ in\ 1\ hour)$
- Is when $X_1 = 0$ and $X_2 = 0$, in which case $\hat{\theta}$ = 1.
- In any other case, the estimator yields a non-probability.
- Which means $\hat{\theta}$ is almost always really absurd even though it's unbiased.

Absurdity Can Be More Subtle

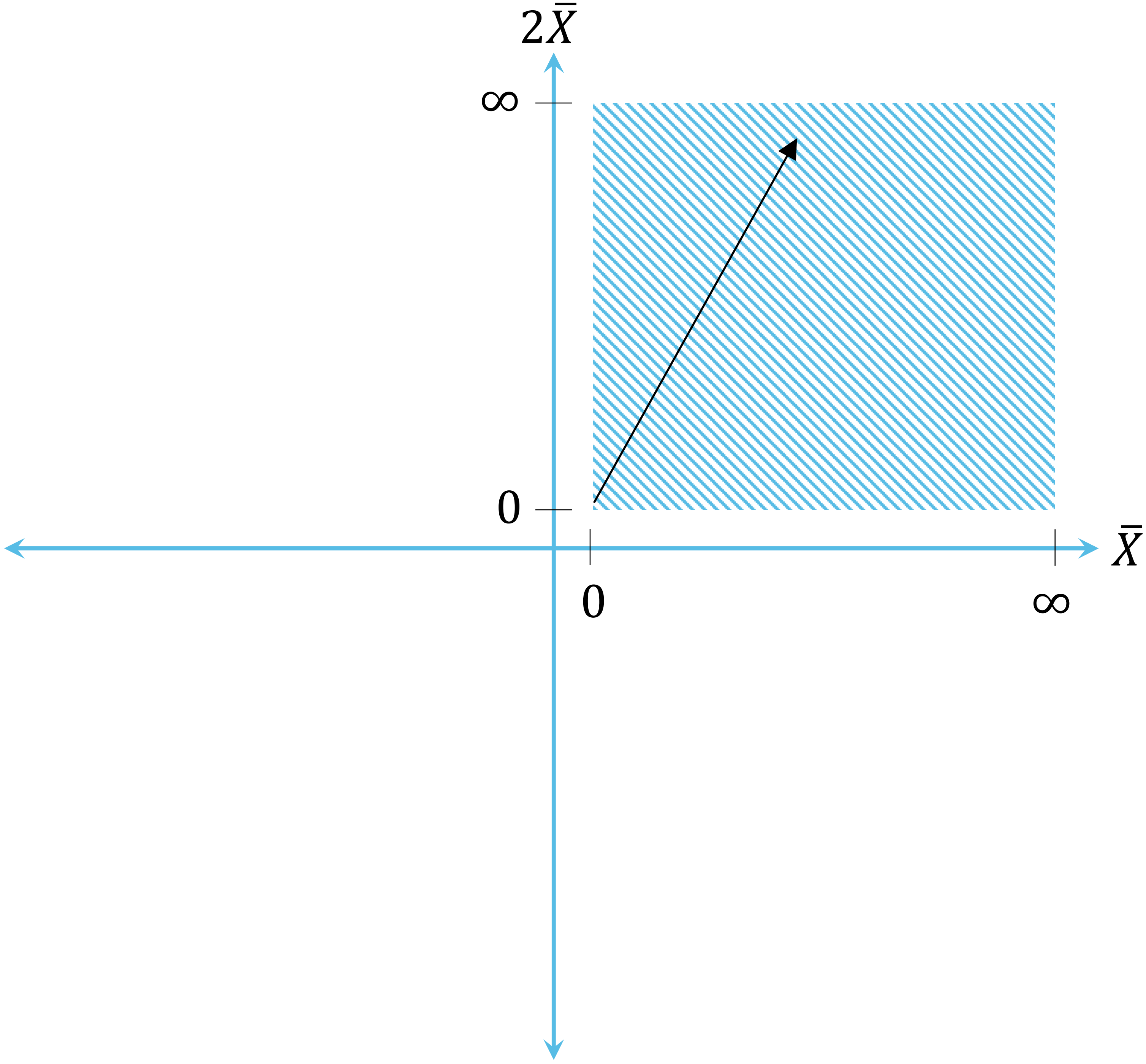
Let $X \sim U(0, \theta)$ with a sample $X_1, X_2, \ldots, X_n$:

e.g., $X =$ Wait time for a bus.

- We want to find the longest possible wait time θ.
- We know that there exist two unbiased estimators for θ:
  - $\hat{\theta} = 2\bar{X}$
  - $\hat{\theta} = \frac{n+1}{n} \, Max \, \{X_1, X_2, \ldots, X_n\}$

Let's Focus On $\hat{\theta} = 2\bar{X}$

- What is the parameter space for θ?
- The parameter space for θ is simply $\Omega_\theta = [0, \infty)$



$2\bar{X}$
∞
0
$\bar{X}$
0
∞

Let's Focus On $\hat{\theta} = 2\bar{X}$

- What is the parameter space for θ?
- The parameter space for θ is simply $\Omega_\theta = [0, \infty)$
- So, $\hat{\theta} = 2\bar{X}$ is not potentially absurd right?
- Consider the following sample:

  {2.5, 4, 0.8, 9, 1.2, 0.5, 3}

- $\bar{X} = \frac{21}{7} = 3$, which yields an estimate of $\hat{\theta} = 2 \cdot 3 = 6$

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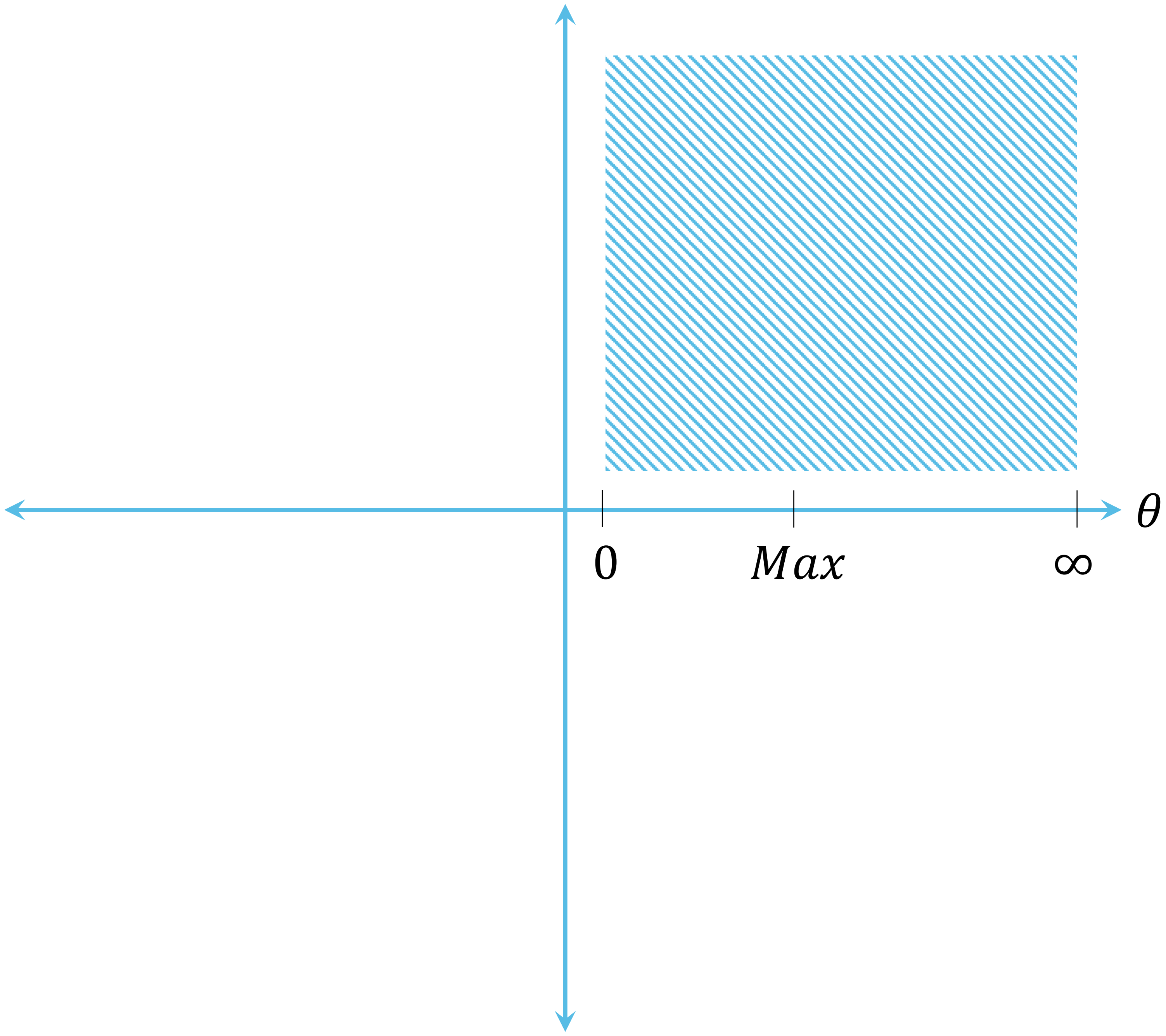

θ
0
Max
∞

Definition: Conditional Potential Absurdi

- An estimator $\hat{\theta}$ is called "Conditionally Potentially Absurd" if, given a sample, there exists a probability of an estimate outside Ω_θ,

  i.e., $P\left(\hat{\theta} \notin \Omega_{\theta|\boldsymbol{X_1},\boldsymbol{X_2}, \dots, \boldsymbol{X_n}}\right) > 0$

- In our example, $X \sim U(0, \theta)$
- $\hat{\theta} = 2\bar{X}$ is conditionally potentially absurd

  i.e, $P\left(\hat{\theta} \notin \Omega_\theta \,|Max(\boldsymbol{X_1}, \boldsymbol{X_2}, \dots, \boldsymbol{X_n})\right) > 0$

Conclusion

We feel that absurdity is a good way to introduce additional properties of estimators to students of statistics. Since unbiasedness is predominant in the undergraduate literature, it should be pointed out that not all unbiased estimators are good estimators. Sometimes they are absurd.

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- Metropolitan State University of Denver
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